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Nonlinear Guidance System for Descent Trajectories

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A survey paper on space rendezvous appears in Astronautica Acta[1] with many references. Terminal guidance for space rendezvous has been divided into two classes: That based on orbital mechanics[2] in the time domain and that based on proportional navigation.[3,4] This paper deals mainly with problems of descent trajectories including soft-landing.[5,6,7,8]

(1) General Equations of Motion

A point mass m is attracted to a planet by a central force proportional to the inverse square of the distance. The equations of motion are[9]

$$\frac{dk}{d\theta} = \frac{2a_\theta}{u^3} \tag{1}$$

$$\frac{d^2u}{d\theta^2} + \frac{\frac{dk}{d\theta}}{2k}\frac{du}{d\theta} + u - \frac{g_o}{ku_o^2} = -\frac{a_r}{ku^2} \tag{2}$$

where $u = \frac{1}{r}$ = inverse of the radius vector, (3)

$k = h^2 = [r^2\dot{\theta}]^2$ = square of the specific angular momentum, (4)

a_θ, a_r = transverse and radial specific forces, respectively,

and g_o = gravitational acceleration at the surface of the planet.

A further transformation is advantageous by virtue of equation (1). Let

$$\frac{du}{d\theta} = \frac{dk}{d\theta}\frac{du}{dk} = \left(\frac{2a_\theta}{u^3}\right)\frac{du}{dk}\ , \tag{5}$$

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If equations (5) and its derivative with respect to θ are substituted into equation (2), we have

$$\frac{d^2u}{dk^2} + \left[\frac{u^3}{2a_\theta}\frac{d}{dk}\left(\frac{2a_\theta}{u^3}\right) + \frac{1}{2k}\right]\frac{du}{dk} + \left(\frac{u^3}{2a_\theta}\right)^2\left[u - \frac{g_o}{ku_o{}^2} + \frac{a_r}{ku^2}\right] = 0, \quad (6)$$

and $\frac{d\theta}{dk} = \frac{u^3}{2a_\theta}$ by inverting equation (1). (7)

The dependent variable k in equation (1) becomes the independent variable k in equations (6) and (7). Equation (6) is a key equation which does not contain θ explicitly. It is linear in u and can be solved independently of equation (7), provided that $\frac{a_\theta}{u^3}$ and $\frac{a_r}{u^2}$ are expressed in terms of k, such as

$$a_\theta = u^3\, G_\theta(k) \text{ and } a_r = u^2\, G_r(k), \quad (8)$$

where $G_\theta(k)$ and $G_r(k)$ are functions of k only.

(2) The Descent Trajectory

A vehicle in orbit is traveling in the direction b'- b as shown on Fig. 1. At point b the vehicle begins its descent phase by firing a retro-rocket having a specific force with a radial component a_r and transverse component a_θ. The descent trajectory is defined by its polar coordinates (r,θ) measured from the point of soft-landing O.

The measurement of θ can be made by inertial means, e.g. an inertial guidance package with both gyros and accelerometers to determine the local vertical with respect to site vertical. The quantity $u = \frac{1}{r}$ can be determined by measuring the altitude of the vehicle above the planet. The controlled specific forces a_θ and a_r will be in terms of the known quantities θ and u.

For the virgin landing of a vehicle near an unexplored planet, it is desirable to allow the astronaut to hover the vehicle near the surface. The conditions of hovering are zero velocities and zero accelerations in both the transverse and radial directions.

(a) Transverse Specific Force

In order to obtain zero transverse velocity the specific angular momentum h_o at point O should be zero, thus $k_o = h_o^2 = 0$. The transverse specific force a_θ should be zero so that the acceleration of the vehicle in the same direction is also zero. Thus we have

$$a_\theta = \frac{u^3}{2\beta} \left(\frac{k}{k_b}\right)^n , \qquad 0 < n \tag{9}$$

where β is a constant. The power index n should be a positive real number so that a_θ is bounded at $k = 0$.

(b) Specific Angular Momentum

If the result of $\frac{u^3}{2a_\theta}$ from equation (9) is substituted into equation (7) and the integration is performed, we have

$$\frac{\theta}{\theta_b} = \frac{k^{1-n}}{{k_b}^{1-n}} , \tag{10}$$

where k_b is the value of k at $\theta = \theta_b$. Solving for k in terms of θ for $n = \frac{1}{2}$ we obtain

$$\frac{k}{k_b} = \frac{\theta^2}{\theta_b^2} . \tag{11}$$

The specific angular momentum ratio $\frac{h}{h_b}$ may be obtained by combining equations (4) and (11) into a linear relation

$$\frac{h}{h_b} = \frac{\theta}{\theta_b} . \tag{12}$$

(c) Radial Specific Force

The differentiation of $\frac{a_\theta}{u^3}$ from equation (9)(using $n = \frac{1}{2}$) with respect to k may be obtained and substituted into equation (6),

$$\frac{d^2u}{dk^2} + (\frac{1}{k}) \frac{du}{dk} + \beta^2(\frac{k_b}{k})(u - \frac{g_o}{ku_o^2} + \frac{a_r}{ku^2}) = 0. \quad (13)$$

The value of a_r must be chosen such that it is bounded everywhere during the course of landing, also it must satisfy the boundary condition of soft landing and hovering. Thus

$$a_r = u^2[\frac{g_o}{u_o^2} - ku + \frac{\xi^2}{k_b}(k^2 - \zeta^2)(u-u_o)] \quad (14)$$

where ξ and ζ are positive real constants.

The value of a_r in equation (14) is bounded since the terms u and k are always positive real finite quantities. At the surface of the planet the value of a_r should be equal to g_o. This is obtainable by substituting $k = 0$ and $u = u_o$ into equation (14). The above relation guarantees that the radial acceleration is zero at the point of landing.

(3) Solution of the Problem

If equation (14) is substituted into equation (13) one obtains a linear equation in u.

$$\frac{d^2u}{dk^2} + (\frac{1}{k}) \frac{du}{dk} + \frac{\beta^2\xi^2}{k^2} (k^2 - \zeta^2)(u-u_o) = 0, \quad (15)$$

or

$$\frac{d^2U}{dk^2} + \frac{1}{k} \frac{dU}{dk} + (\omega^2 - \frac{\omega^2\zeta^2}{k^2}) U = 0 , \quad (16)$$

where

$$\dot{\omega} = \beta\xi, \quad (17)$$

and

$$U = -(u - u_o) = \frac{r-r_o}{rr_o} . \quad (18)$$

Under the boundary condition that $U = 0$ at $k = 0$ the solution of equation (16) in term of fractional order Bessel Functions of the first kind as shown in Fig. 2(a) is [10,11,12]

$$\frac{U}{U_b} = \frac{J_{\omega\zeta}(\omega k_b \frac{\theta^2}{\theta_b^2})}{J_{\omega\zeta}(\omega k_b)} \quad , \tag{19}$$

where U_b is the value of U at $\theta = \theta_b$.

The derivative of U with respect to k can be shown to be

$$\frac{J_{\omega\zeta}(\omega k_b)}{\omega U_b} \frac{dU}{dk} = \frac{\zeta}{k_b}(\frac{\theta}{\theta_b})^{-2} J_{\omega\zeta}(\omega k_b \frac{\theta^2}{\theta_b^2}) - J_{\omega\zeta+1}(\omega k_b \frac{\theta^2}{\theta_b^2}) \ . \tag{20}$$

For the Bessel Function of 1/2 order equations (19) and (20) become simply sine function of θ^2

$$\frac{U}{U_b} = (\frac{\theta}{\theta_b})^{-1} \frac{\sin(\omega k_b \frac{\theta^2}{\theta_b^2})}{\sin(\omega k_b)} \qquad \text{and} \tag{21}$$

$$\frac{\sin \omega k_b}{\omega U_b} (\frac{dU}{dk}) = (\frac{\theta}{\theta_b})^{-1} [\cos(\omega k_b \frac{\theta^2}{\theta_b^2}) - \frac{1}{2}(\omega k_b \frac{\theta^2}{\theta_b^2})^{-1} \sin(\omega k_b \frac{\theta^2}{\theta_b^2})] . \tag{22}$$

By differentiating equation (19) with respect to θ one obtains

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b \ U_b} \frac{dU}{d\theta} = \frac{\theta}{\theta_b}[\frac{\zeta}{k_b}(\frac{\theta}{\theta_b})^{-2} J_{\omega\zeta}(\omega k_b \frac{\theta^2}{\theta_b^2}) - J_{\omega\zeta+1}(\omega k_b \frac{\theta^2}{\theta_b^2})] . \tag{23}$$

For $0 < \omega\zeta$, the term $J_{\omega\zeta+1}(0)$ is zero. However, the term $J_{\omega\zeta}$ is of the order of magnitude of $(\theta^2)^{\omega\zeta}$ as $\theta \to 0$. Therefore the term $\frac{dU}{d\theta}$ is of the order of

$$\theta^{1-2+2\omega\zeta} \doteq \theta^{2\omega\zeta-1} .$$

From equation (24) and the condition $\frac{dr}{d\theta} = \frac{1}{u^2}\frac{dU}{d\theta}$, where u^2 is real and positive it is concluded that as $\theta \to 0$,

$\frac{dU}{d\theta} \to \infty$ for $\quad 0 < \omega\zeta < \frac{1}{2}$, also $\frac{dr}{d\theta} \to \infty$, vertical landing;

$\frac{dU}{d\theta}$ = constant for $\omega\zeta = \frac{1}{2}$, also $\frac{dr}{d\theta}$ = constant, inclined landing ;

$$\frac{dU}{d\theta} = 0 \quad \text{for } \frac{1}{2}\ \omega\zeta, \quad \text{also } \frac{dr}{d\theta} = 0, \text{ horizontal landing.} \tag{25}$$

(4) The Characteristic Root

The state of the vehicle at the starting point b is completely defined if the quantities k_b, θ_b, U_b and $\left.\frac{dU}{d\theta}\right|_b = W_b$ with respect to the planet are known. From equation (23) we have

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b\ U_b} W_b = \frac{\omega\zeta}{\omega k_b}\ J_{\omega\zeta}(\omega k_b) - J_{\omega\zeta+1}(\omega k_b)\ . \tag{26}$$

The above equation may be solved for its characteristic root

$$\omega k_b = \eta \tag{27}$$

provided θ_b, U_b, W_b and $\omega\zeta$ are given. It is emphasized here that η is the lowest positive real root of equation (26).

An example is given here for descent from a circular orbit ($W_b = 0$) along a trajectory of Bessel Function of 1/2 order. Thus from equation (23) one obtains

$$\frac{\theta_b \sin\ (\omega k_b)}{U_b \omega k_b} \left.\left(\frac{dU}{d\theta}\right)\right|_{\theta=\theta_b} = 2\ \cos\left(\omega k_b\ \frac{\theta^2}{\theta_b^2}\right) - \left.\left(\omega k_b\ \frac{\theta^2}{\theta_b^2}\right)^{-1} \sin\left(\omega k_b\ \frac{\theta^2}{\theta_b^2}\right)\right|_{\theta=\theta_b} =0 \tag{28}$$

from which we have

$$\tan\ \omega k_b = 2\ \omega k_b\ . \tag{29}$$

The value $\omega k_b = \eta = 1.16562$ may be obtained for its fundamental root for the case $\omega\zeta = 1/2$.

The dimensionless form of $\frac{dr}{d\theta}$ is shown in Fig. 2(b) for $\omega k_b = \eta$ and various values of $\omega\zeta$.

(5) The Angular and Radial Velocities

With the aid of equations (4) and (11) the angular velocity $\dot{\theta}$ may be written as

$$\frac{\dot{\theta}}{\dot{\theta}_b} = \frac{\theta}{\theta_b}\frac{u^2}{u_b^2} . \tag{30}$$

From equations (18), (19) and (30) we have

$$\frac{\dot{\theta}}{\dot{\theta}_b} = \left(\frac{u_o}{u_b}\right)^2\left(\frac{\theta}{\theta_b}\right)\left[1 - \frac{U_b}{u_o}\frac{J_{\omega\zeta}(\omega k_b \frac{\theta^2}{\theta_b^2})}{J_{\omega\zeta}(\omega k_b)}\right]^2 . \tag{31}$$

The value of $\dot{\theta}$ at $\theta = 0$ is zero as shown in Fig. 3(a). This is in conformity with the requirement of soft landing. Also

$$\frac{dr}{dt} = u^{-2}\frac{dU}{dt} , \tag{32}$$

where
$$\frac{dU}{dt} = \frac{d\theta}{dt}\frac{dU}{d\theta} = k^{1/2} u^2 \frac{dU}{d\theta} . \tag{33}$$

Combining equations (11), (23), (32) and (33) one obtains

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b U_b}\frac{dr}{dt} = \left(\frac{\theta}{\theta_b}\right)^2\left[\frac{\zeta}{k_b}\left(\frac{\theta}{\theta_b}\right)^{-2} J_{\omega\zeta}(\omega k_b \frac{\theta^2}{\theta_b^2}) - J_{\omega\zeta+1}(\omega k_b \frac{\theta^2}{\theta_b^2})\right], \tag{34}$$

which is also plotted in Fig. 3(b) for $\omega k_b = \eta$ and various values of $\omega\zeta$.

It is expected that the reasoning given in discussing equation (24) holds also here. Thus the order of magnitude of the term $\frac{dr}{dt}$ is

$$\theta^{2-2+2\omega\zeta} = \theta^{2\omega\zeta} \tag{35}$$

therefore
$$\left.\frac{dr}{dt}\right|_{\theta=0} = 0, \quad \text{for } 0 < \omega\zeta . \tag{36}$$

Equation (36) indicates that the radial velocity of the vehicle is also zero at the point of contact on the surface of a planet as would be expected.

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